

Finitary incidence algebras

N. S. Khripchenko, B. V. Novikov

Dept of Mechanics and Mathematics, Kharkov National University, Ukraine
e-mail: boris.v.novikov@univer.kharkov.ua

It is well known, that incidence algebras can be defined only for locally finite partially ordered sets [2, 4]. At the same time, for example, the poset of cells of a noncompact cell partition of a topological space is not locally finite. On the other hand, some operations, such as the order sum and the order product [4], do not save the locally finiteness. So it is natural to try to generalize the concept of incidence algebra.

In this article we consider the functions in two variables on an arbitrary poset (*finitary series*), for which the convolution operation is defined. We obtain the generalization of incidence algebra — *finitary incidence algebra* and describe its properties: invertibility, the Jacobson radical, idempotents, regular elements. As a consequence a positive solution of the isomorphism problem for such algebras is obtained.

1 Definition of finitary algebras

In what follows K denotes a fixed field, P an arbitrary partially ordered set (poset). Denote by $I(P)$ the set of formal sums of the form

$$\alpha = \sum_{x \leq y} \alpha(x, y)[x, y] \quad (1)$$

where $x, y \in P$, $\alpha(x, y) \in K$, $[x, y] = \{z \in P \mid x \leq z \leq y\}$. In the general case $I(P)$ is not an algebra, so we call it *an incidence space*.

A formal sum (1) is called *a finitary series* if for any $x, y \in P$, $x < y$, there is only a finite number of subsegments $[u, v] \subset [x, y]$ such that $u \neq v$ and $\alpha(u, v) \neq 0$. The set of all finitary series is denoted by $FI(P)$.

If P is not locally finite then, as we mentioned above, the usual multiplication (convolution) of the series

$$\alpha\beta = \sum_{x \leq y} \alpha(x, y)[x, y] \cdot \sum_{u \leq v} \beta(u, v)[u, v] = \sum_{x \leq y} \left(\sum_{x \leq z \leq y} \alpha(x, z)\beta(z, y) \right) [x, y]$$

is not always defined in $I(P)$. The following proposition shows that in the general situation it is reasonable to consider $FI(P)$ instead of $I(P)$.

Theorem 1 *$FI(P)$ is an associative algebra and $I(P)$ is a module over it.*

Proof. Obviously, it is sufficient to prove that if $\alpha \in FI(P)$, $\beta \in I(P)$ then $\alpha\beta$ is defined, and if in addition $\beta \in FI(P)$ then $\alpha\beta \in FI(P)$.

So let $\alpha \in FI(P)$, $\beta \in I(P)$, $x, y \in P$ and $x \leq y$. Then the sum

$$\gamma(x, y) = \sum_{x \leq z \leq y} \alpha(x, z)\beta(z, y)$$

contains only a finite number of nonzero values $\alpha(x, z)$. Since every $\alpha(x, z)$ appears in the sum at most one time, this sum is finite. Therefore $\alpha\beta$ is defined.

Now let $\alpha, \beta \in FI(P)$. Suppose that $\gamma = \alpha\beta \notin FI(P)$. This means that we can find such $x, y \in P$, $x \leq y$, that there is an infinite number of subsegments $[u_i, v_i] \subset [x, y]$ ($i = 1, 2, \dots$), for which $u_i \neq v_i$ and $\gamma(u_i, v_i) \neq 0$. At least one of the sets $\{u_i\}$, $\{v_i\}$ must be infinite; for example, let $|\{u_i\}| = \infty$.

It follows from $\gamma(u_i, v_i) \neq 0$ that for each i there is $z_i \in [u_i, v_i]$ such that $\alpha(u_i, z_i) \neq 0 \neq \beta(z_i, v_i)$. Since $\alpha \in FI(P)$ and $[u_i, z_i] \subset [x, y]$, we have $u_i = z_i$ for an infinite number of indexes. But then $\beta(u_i, v_i) \neq 0$ for this set of indexes, what is impossible, since $u_i \neq v_i$. ■

Example. Let \mathbb{N} be the set of positive integers with the natural order, $\overline{\mathbb{N}}$ its isomorphic copy and $P = \mathbb{N} \oplus \overline{\mathbb{N}}$ their order sum (i.e. $a < \overline{b}$ for each $a \in \mathbb{N}$, $\overline{b} \in \overline{\mathbb{N}}$).

The algebra $FI(\overline{\mathbb{N}})$ embeds into $FI(P)$; it is sufficient for this to extend each series from $FI(\overline{\mathbb{N}})$ by zero values. This is not the case for $FI(\mathbb{N})$: since each segment of the form $[a, \overline{b}]$ ($a \in \mathbb{N}$, $\overline{b} \in \overline{\mathbb{N}}$) is infinite, each series from $FI(P)$ is nonzero only at a finite number of the segments from \mathbb{N} . Therefore $FI(P)$ contains not $FI(\mathbb{N})$, but the algebra of *finite* formal sums of the segments from \mathbb{N} .

2 Properties of finitary algebras

Denote by δ the identity of the algebra $FI(P)$. Then $\delta(x, y) = \delta_{xy}$ where δ_{xy} is the Kronecker delta.

Theorem 2 *A series $\alpha \in FI(P)$ is invertible iff $\alpha(x, x) \neq 0$ for each $x \in P$. Moreover $\alpha^{-1} \in FI(P)$.*

Proof. *Necessity.* Let $\alpha\beta = \delta$. Then $\alpha(x, x)\beta(x, x) = \delta(x, x) = 1$, so $\alpha(x, x) \neq 0$.

Sufficiency. Let $\alpha(x, x) \neq 0$ for all $x \in P$ and $[u, v]$ be a segment from P , $u \neq v$. A series β , which is inverse for α , exists if $\beta(x, x) = \alpha(x, x)^{-1}$ and

$$\beta(u, v) = -\alpha(u, u)^{-1} \sum_{u < x \leq v} \alpha(u, x)\beta(x, v) \quad (2)$$

(note that the sum on the right-hand side is defined since $\alpha(u, x)$ is different from zero only for a finite number of elements $x \in [u, v]$). We prove that a solution of the equation (2) exists and can be computed recursively in a finite number of steps.

Denote by $C_\alpha(u, v)$ the number of subsegments $[x, y] \subseteq [u, v]$ such that $x \neq y$ and $\alpha(x, y) \neq 0$. By the definition of finitary series $C_\alpha(u, v)$ is finite. We shall prove our assertion by induction on $C_\alpha(u, v)$.

If $C_\alpha(u, v) = 0$ then $\beta(u, v) = 0$. If $C_\alpha(u, v) = 1$ and $u < x_0 \leq v$, $\alpha(u, x_0) \neq 0$ then

$$\begin{aligned} \beta(u, v) &= -\alpha(u, u)^{-1}\alpha(u, x_0)\beta(x_0, v) \\ &= \begin{cases} -\alpha(u, u)^{-1}\alpha(u, x_0)\alpha(v, v)^{-1}, & \text{if } x_0 = v, \\ 0, & \text{if } x_0 \neq v \text{ (since } C_\alpha(x_0, v) = 0). \end{cases} \end{aligned}$$

Now suppose that $\beta(x, y)$ is defined for all x, y such that $C_\alpha(x, y) < n$. Let $C_\alpha(u, v) = n$. If $\alpha(u, x)$ is nonzero for some x , $u < x \leq v$, then $C_\alpha(x, v) \leq n - 1$ and, by the induction hypothesis, $\beta(x, v)$ is defined. Thus, every summand from the right-hand side of (2) is defined, and since the sum is finite, $\beta(u, v)$ is also defined.

Suppose that $\alpha^{-1} \notin FI(P)$. Then we can find a segment $[x, y]$ and an infinite number of subsegments $[u_i, v_i] \subset [x, y]$, $u_i \neq v_i$, for which $\alpha^{-1}(u_i, v_i) \neq 0$. From the equalities $\alpha^{-1}\alpha = \alpha\alpha^{-1} = \delta$ we have:

$$\alpha^{-1}(u_i, v_i) = -\alpha(v_i, v_i)^{-1} \sum_{u_i \leq z_i < v_i} \alpha^{-1}(u_i, z_i)\alpha(z_i, v_i), \quad (3)$$

$$\alpha^{-1}(u_i, v_i) = -\alpha(u_i, u_i)^{-1} \sum_{u_i < z_i \leq v_i} \alpha(u_i, z_i) \alpha^{-1}(z_i, v_i). \quad (4)$$

It follows from (3) that for each i there is z_i such that $\alpha(z_i, v_i) \neq 0$. But the number of such segments is finite, therefore $|\{v_i\}| < \infty$. Similarly $|\{u_i\}| < \infty$ from (4), a contradiction. Hence $\alpha^{-1} \in FI(P)$. ■

Corollary 1 *Left, right and two-sided invertibilities in $FI(P)$ coincide.*

Proof. If, for example, α is right invertible, then, as was shown in the proof of necessity, $\alpha(x, x) \neq 0$ for all $x \in P$. Now it follows from the theorem, that α is two-sided invertible. ■

In what follows $\text{Rad } A$ denotes the Jacobson radical of algebra A .

Corollary 2 $\alpha \in \text{Rad } FI(P)$ iff $\alpha(x, x) = 0$ for all $x \in P$.

Proof. As in [2], the proof follows from [1], Proposition 1.6.1. ■

Corollary 3 *The factor algebra $FI(P)/\text{Rad } FI(P)$ is commutative.* ■

The following proposition describes the idempotents of a finitary algebra.

We call an element $\alpha \in FI(P)$ *diagonal* if $\alpha(x, y) = 0$ for $x \neq y$. It is obvious that a diagonal element α is idempotent iff $\alpha(x, x)$ is equal to 0 or 1 for all $x \in P$.

Theorem 3 *Each idempotent $\alpha \in FI(P)$ is conjugate to the diagonal idempotent ε , such that $\varepsilon(x, x) = \alpha(x, x)$ for all $x \in P$.*

Proof. According to the corollary 2, $\rho = \alpha - \varepsilon \in \text{Rad } FI(P)$. Since $\alpha^2 = \alpha$, we have:

$$\varepsilon\rho + \rho\varepsilon = \rho - \rho^2 \quad (5)$$

Multiplying this equality by ε on the left, we obtain:

$$\varepsilon\rho\varepsilon + \varepsilon\rho^2 = 0 \quad (6)$$

Set $\beta = \delta + (2\varepsilon - \delta)\rho$ where δ is the identity of $FI(P)$. Since $\rho \in \text{Rad } FI(P)$, the series β is invertible by theorem 2.

By (6), we have:

$$\begin{aligned}\beta\alpha &= (\delta + 2\varepsilon\rho - \rho)(\varepsilon + \rho) = \varepsilon - \rho\varepsilon + \rho - \rho^2, \\ \varepsilon\beta &= \varepsilon(\delta + 2\varepsilon\rho - \rho) = \varepsilon + \varepsilon\rho.\end{aligned}$$

From (5) we obtain: $\beta\alpha = \varepsilon\beta$. Hence $\alpha = \beta^{-1}\varepsilon\beta$. ■

An element α of the algebra A is called *regular* if there is $\chi \in A$, such that $\alpha\chi\alpha = \alpha$.

Theorem 4 *For each regular $\alpha \in FI(P)$ there are a diagonal idempotent $\varepsilon \in FI(P)$ and invertible elements $\beta, \gamma \in FI(P)$, such that $\alpha = \beta\varepsilon\gamma$.*

Proof. The sufficiency is obvious — we can set $\chi = \gamma^{-1}\varepsilon\beta^{-1}$. Let us prove the necessity.

Evidently, $\alpha\chi$ and $\chi\alpha$ are idempotents. By the theorem 3, there is an invertible $\eta \in FI(P)$, such that $\alpha\chi = \eta^{-1}\varepsilon\eta$, where ε is a diagonal idempotent, and, moreover, $\varepsilon(x, x) = \alpha(x, x)\chi(x, x)$. In consequence of the regularity, the last equality is equivalent to the statement

$$\varepsilon(x, x) = 0 \iff \alpha(x, x) = 0. \quad (7)$$

Similarly, $\chi\alpha = \gamma^{-1}\varepsilon_1\gamma$ for some invertible $\gamma \in FI(P)$ and diagonal idempotent $\varepsilon_1 \in FI(P)$. In fact $\varepsilon_1 = \varepsilon$ since (7) holds for ε_1 too.

From the regularity of α we have:

$$\alpha = \alpha\chi\alpha = \alpha\chi\alpha\chi\alpha = \eta^{-1}\varepsilon\eta\alpha\gamma^{-1}\varepsilon\gamma.$$

Since η and γ are invertible, it follows from (7) that

$$\eta\alpha\gamma^{-1}(x, x) = 0 \iff \varepsilon(x, x) = 0.$$

Therefore the element $\eta\alpha\gamma^{-1}$ can be rewritten as $\eta\alpha\gamma^{-1} = \eta_1\varepsilon + \rho$ where η_1 is diagonal invertible and $\rho \in \text{Rad } FI(P)$. Since diagonal elements commute, $\varepsilon\eta\alpha\gamma^{-1}\varepsilon = \eta_1\varepsilon + \varepsilon\rho\varepsilon = (\eta_1 + \varepsilon\rho)\varepsilon$, and, moreover, $\eta_1 + \varepsilon\rho$ is invertible by the theorem 2. Thus,

$$\alpha = \eta^{-1}\varepsilon\eta\alpha\gamma^{-1}\varepsilon\gamma = \eta^{-1}(\eta_1 + \varepsilon\rho)\varepsilon\gamma. \quad \blacksquare$$

In conclusion we consider a property of elements, which is intermediate between the invertibility and the regularity.

Let α be a regular element, $\alpha\chi\alpha = \alpha$. Then the element $\alpha^* = \chi\alpha\chi$ satisfies the equations:

$$\alpha\alpha^*\alpha = \alpha, \quad \alpha^*\alpha\alpha^* = \alpha^*. \quad (8)$$

We call α *superregular* if there exists only one α^* for which the equations (8) are fulfilled. For instance, invertible and zero elements are superregular. It turns out that in incidence algebras superregular elements can be described with the help of invertible ones:

Corollary 4 *Let $P = \bigcup_{i \in I} P_i$ be decomposition of P into a disjoint union of connected components and $U(P_i)$ be the group of invertible elements of the algebra $FI(P_i)$. Then the set of superregular elements coincides with the sum of semigroups $\bigoplus_{i \in I} U(P_i)^0$.*

Proof. It is easy to see that $FI(P) = \bigoplus_{i \in I} FI(P_i)$. Obviously, each superregular element $\alpha \in FI(P)$ can be represented as the sequence $(\alpha_i)_{i \in I}$ of the superregular elements $\alpha_i \in FI(P_i)$ and, moreover, $\alpha^* = (\alpha_i^*)_{i \in I}$. The statement will be proven if we shall show that $\alpha_i \in U(P_i)^0$.

Suppose the contrary. First note that the multiplication on the left or on the right by an invertible element preserves superregularity. Therefore, by the theorem 4, we can consider α_i to be a diagonal idempotent. By assumption, there is a pair $x, y \in P_i$, such that $\alpha_i(x, x) \neq \alpha_i(y, y)$, and, moreover, we can choose it in such a way, that $x < y$ since P_i is connected. Define $\alpha_i^* = \alpha_i + \rho$, where

$$\rho(u, v) = \begin{cases} 1, & \text{if } u = x, v = y, \\ 0, & \text{otherwise.} \end{cases}$$

By the direct checking, we make sure, that the equalities (8) hold for α_i , and, since α_i is a superregular idempotent, $\alpha_i^* = \alpha_i$. This contradicts the definition of ρ . ■

3 The isomorphism problem

It is well known [2] that the isomorphism problem for locally finite posets is solved positively: if P and Q are locally finite and $I(P) \cong I(Q)$, then $P \cong Q$

(in the case K is a ring, an answer can be negative, see, for example, [3]). In this section we consider the isomorphism problem for finitary series.

Recall, that an idempotent $\alpha \neq 0$ is *primitive*, if $\alpha\varepsilon = \varepsilon\alpha = \alpha$ for some idempotent ε implies that ε is equal to 0 or α .

We will need the idempotents of special form δ_x^P , defined for an arbitrary element $x \in P$ as follows:

$$\delta_x^P(u, v) = \begin{cases} 1, & \text{if } u = v = x, \\ 0, & \text{if } u \neq x \text{ or } v \neq x. \end{cases} \quad (9)$$

Lemma 1 *An idempotent α is primitive iff it is conjugate to δ_x^P for some $x \in P$.*

Proof. The conjugation, being an automorphism, preserves the primitivity. So by the theorem 3, it is sufficient to prove that the diagonal primitive idempotents are nothing but δ_x^P .

1) We first show that δ_x^P is primitive. Indeed, let $\alpha\delta_x^P = \alpha$ holds for some idempotent α . Then for each segment $[u, v] \subset P$ we obtain:

$$\alpha(u, v)\delta_x^P(v, v) - \alpha(u, v) = 0,$$

i. e. $\alpha(u, v) = 0$ if $v \neq x$.

Similarly $\delta_x^P\alpha = \alpha$ is equivalent to $\alpha(u, v) = 0$ for $u \neq x$. If $\alpha(x, x) = 0$ then we get $\alpha \equiv 0$. On the other hand, if $\alpha(x, x) = 1$ then $\alpha = \delta_x^P$.

2) Let ε be a diagonal primitive idempotent.

Since $\varepsilon \neq 0$, there is such $x \in P$ that $\varepsilon(x, x) = 1$. Consider the idempotent δ_x^P . Obviously, the equalities $\varepsilon\delta_x^P = \delta_x^P = \delta_x^P\varepsilon$ hold for it. Therefore $\delta_x^P = \varepsilon$, since ε is primitive and $\delta_x^P \neq 0$. ■

It is easy to see that if δ_x^P and δ_y^P are conjugate then $x = y$.

Theorem 5 *Let P and Q be arbitrary posets. Then*

$$FI(P) \cong FI(Q) \implies P \cong Q$$

Proof. Let $\Phi : FI(P) \rightarrow FI(Q)$ be an isomorphism. For each $x \in P$ the image $\Phi(\delta_x^P)$ of δ_x^P is primitive and, by lemma 1, is conjugate to the idempotent δ_y^Q for some $y \in Q$. According to the remark before theorem, the

element y is defined uniquely. Thus, Φ generates the bijection $\varphi : P \rightarrow Q$, such that $\Phi(\delta_x^P)$ is conjugate to $\delta_{\varphi(x)}^Q$.

Let us prove that φ preserves the order.

It is easy to see that for all $x, y \in P$

$$x \leq y \iff \delta_x^P FI(P) \delta_y^P \neq 0.$$

Similarly for all $u, v \in Q$

$$u \leq v \iff \delta_u^Q FI(Q) \delta_v^Q \neq 0.$$

Let $x \leq y$ and $\Phi(\delta_x^P) = \beta_1 \delta_{\varphi(x)}^Q \beta_1^{-1}$, $\Phi(\delta_y^P) = \beta_2 \delta_{\varphi(y)}^Q \beta_2^{-1}$ for some invertible elements β_1 and β_2 . From the bijectivity of Φ we obtain

$$\Phi(\delta_x^P) FI(Q) \Phi(\delta_y^P) = \Phi(\delta_x^P FI(P) \delta_y^P) \neq 0.$$

Hence

$$\delta_{\varphi(x)}^Q FI(Q) \delta_{\varphi(y)}^Q = \beta_1^{-1} \Phi(\delta_x^P) \beta_1 FI(Q) \beta_2^{-1} \Phi(\delta_y^P) \beta_2 = \beta_1^{-1} \Phi(\delta_x^P) FI(Q) \Phi(\delta_y^P) \beta_2$$

since β_1 and β_2 are invertible. Therefore $\varphi(x) \leq \varphi(y)$. ■

Corollary 5 *Let P be not locally finite. Then there is no locally finite poset Q , such that $FI(P) \cong I(Q)$.* ■

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